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XIV.

ON SOME SIMPLE CASES OF ELECTRIC FLOW IN FLAT CIRCULAR PLATES.

By B. O. Peirce.

Presented May 13, 1891.

1. It is well known * that, if in a thin conducting plate of indefinite extent there are two sources and a sink, each of strength numerically equal to m, situated respectively at points A, B, C, which lie in order upon a straight line, one of the lines of flow consists in part of a circumference of radius $\sqrt{CA \cdot CB}$ drawn around C as a centre; so that the flow inside the circumference would be unchanged if the part of the plate outside it were cut away. In other words, if a circumference be drawn in a thin conducting plane plate of indefinite extent, the "image" in this circumference of a source, of strength m, situated at a point P in the plane, is made up of a sink, of strength m, at the centre of the circle, and a source of the same strength at Q, the inverse point of P with respect to the circumference.

If a sink be regarded as a negative source, it follows from this that if inside a circumference drawn in a thin plane conducting plate of indefinite extent there are sources at the points $A_1, A_2, A_3, \ldots. A_k$, of strengths algebraically equal to $m_1, m_2, m_3, \ldots. m_k$, respectively, and sources of strengths algebraically equal to $-m_1$, $-m_2$, $-m_3$, $\ldots. -m_k$, at the corresponding inverse points, then, if $m_1 + m_2 + m_3 + \ldots. + m_k = 0$, there is no flow of electricity across the circumference.

2. If, at a fixed point P in a thin plane plate there is a sink of strength numerically equal to m, and at another point P' in the plate an equal source, and if P' be made to approach P as a limit and the product $m \cdot \overline{PP'}$ be kept always equal to a given constant, μ, we have

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* Kirchhoff, Pogg. Ann. 1845, p. 497; W. R. Smith, Proc. Ed. Roy. Soc., 1869-70; Foster and Lodge, Phil. Mag., 1875; Minchin's "Uniplanar Kinematics," p. 213; etc., etc.

as a limit a "doublet"* of strength μ, the axis of which is PX, the limiting position of the straight line drawn *from* P *to* P'. We shall find it convenient to represent sources and sinks respectively by black and unshaded circles, and doublets by circles half black and half unshaded. The black portion of a doublet circle indicates the directions in which there is a flow *away from* the point where the doublet is situated; the unshaded portion indicates the directions from which there is a flow *towards* this point. The axis of a doublet bisects both the black and unshaded portions of the doublet circle. (See Fig. 1.) If P be used as origin and PX as axis of abscissas, the velocity potential function due to the doublet is $\phi = -\frac{\mu x}{x^2 + y^2}$ and the flow function is $\psi = \frac{\mu y}{x^2 + y^2}$. If $x + yi = z$, these are respectively the real part and the real factor of the imaginary part of the function $\frac{-\mu}{z}$.

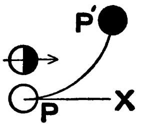

FIG. 1.

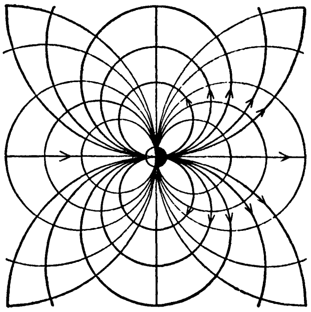

FIG. 2.

The equipotential lines and the lines of flow are circles (see Fig. 2) touching the axes of y and x respectively at the origin.

In uniplanar motion, the velocity at a point M due to a doublet of strength μ at a point P is numerically equal to $\frac{\mu}{\overline{PM}^2}$, and is directed along a straight line MN such that PM bisects the angle between MN and the axis of the doublet. In all that follows, the motion will be assumed to be uniplanar.

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* Basset, Treatise on Hydrodynamics, Art. 47; Mascart and Joubert, Treatise on Electricity and Magnetism, Art. 151; Neumann, Untersuchungen über das Logarithmische und Newton'sche Potential, Chap. IV.; etc., etc.

3. If, in the subjoined figure, $\overline{OL} \cdot \overline{ON} = \overline{OL'} \cdot \overline{ON'} = \overline{OA}^2$, and if there are sources, of strength m, at L' and N', and equal sinks at L and N, there will be no flow across the circumference drawn around O with OA as radius. The value at P, the co-ordinates of which are x and y, of the flow function due to these sources and sinks is, if $OL = a$, $LL' = \delta$, $OA = r$,

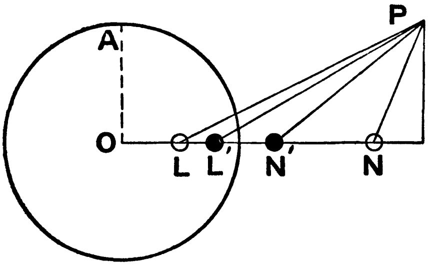

Fig. 3.

$$\psi_P = m\left\{\tan^{-1}\frac{y}{x-(a+\delta)} - \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x-\dfrac{r^2}{a+\delta}} - \tan^{-1}\frac{y}{x-\dfrac{r^2}{a}}\right\}$$

$$= m\left\{\tan^{-1}\frac{\delta y}{(x-a)^2-\delta(x-a)+y^2} - \tan^{-1}\frac{r^2\delta y}{(ax-r^2)^2+\delta x(ax-r^2)+a(a+\delta)y^2}\right\}.$$

If now δ be made to approach zero as a limit, and m to increase in such a way that $m \cdot \delta$ is always equal to a constant, μ, the circumference just mentioned continues to be a stream line and the flow function due to the combination of a doublet of strength μ at L with axis coincident with the radius drawn through L, and the image of this doublet in the circumference is

$$\underset{\delta \doteq 0}{\text{Limit}}\, \psi_P = \mu\left\{\frac{y}{(x-a)^2+y^2} - \frac{r^2}{a^2} \cdot \frac{y}{\left(x-\dfrac{r^2}{a}\right)^2+y^2}\right\}. \qquad (1)$$

It is evident from this result (see section 2) that the image of a doublet of strength μ so situated at a point L — which is at a distance a from the centre of a circumference drawn with radius r on a thin plane indefinitely extended plate — that its axis coincides with the radius drawn through L, is a doublet of strength* $\mu \frac{r^2}{a^2}$ and axis making an angle of 180° with that of the first, at the inverse point of L with respect to the circumference.

It is to be noticed that the flow function given by (1) is the real factor of the imaginary part of the function

$$w = -\mu \left\{ \frac{1}{z-a} - \frac{r^2}{a^2} \cdot \frac{1}{z - \frac{r^2}{a}} \right\}.$$

4. By differentiating (1) partially with respect to y and $-x$, respectively, we get the velocity components at the point (x, y) corresponding to the case which we have been considering. They are

$$u = D_y \psi = \mu \left\{ \frac{(x-a)^2 - y^2}{[(x-a)^2 + y^2]^2} - \frac{r^2}{a^2} \cdot \frac{\left(x - \frac{r^2}{a}\right)^2 - y^2}{\left[\left(x - \frac{r^2}{a}\right)^2 + y^2\right]^2} \right\}; \qquad (2)$$

$$v = -D_x \psi = 2\mu y \left\{ \frac{x-a}{[(x-a)^2 + y^2]^2} - \frac{r^2(ax - r^2)}{a^3 \left[\left(x - \frac{r^2}{a}\right)^2 + y^2\right]^2} \right\}. \qquad (3)$$

If we make a approach zero in order to obtain the flow in a thin circular plate due to a doublet at the centre, ψ as given by (1) grows larger without limit, but u and v approach the definite limits

$$u_0 = \mu \left\{ \frac{x^2 - y^2}{[x^2 + y^2]^2} - \frac{1}{r^2} \right\}, \qquad (4)$$

$$v_0 = \frac{2\mu x y}{[x^2 + r^2]^2}; \qquad (5)$$

and these expressions solve the problem.

This case may be treated in another way, however. In Figure 4 let $OL = OL' = \delta$ and $OM \cdot OL = OL' \cdot OM' = \overline{OA}^2 = r^2$; then, if there are sources of strength m at L and M and equal sinks at L' and M', one of the lines of flow due to this combination will consist in part of the circumference of radius r drawn about O as centre.

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* This does not agree with the statement made in the first edition of Basset's Treatise on Hydrodynamics, p. 56, where there seems to be a typographical error.

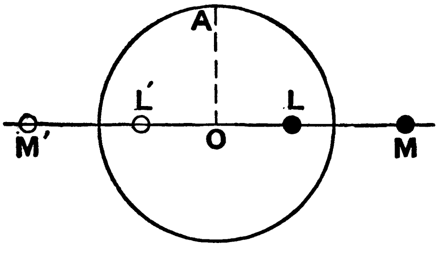

FIG. 4.

The value at the point (x, y) of the flow function is

$$\psi = m\left\{\tan^{-1}\frac{y}{x-\delta} + \tan^{-1}\frac{y}{x-\frac{r^2}{\delta}} - \tan^{-1}\frac{y}{x+\delta} - \tan^{-1}\frac{y}{x+\frac{r^2}{\delta}}\right\}$$

$$= m\left\{\tan^{-1}\frac{2\,\delta\,y}{x^2+y^2-\delta^2} + \tan^{-1}\frac{2\,r^2\,\delta\,y}{\delta^2\,(x^2+y^2)-r^4}\right\}.$$

If, now, δ be made to approach zero as a limit, and m to increase in such a way that $2\,m\,\delta$ is equal to the constant μ,

$$\psi_0 = \underset{\delta = 0}{\text{Limit}}\ \psi = -\frac{\mu\,y}{r^2}\left(\frac{x^2+y^2-r^2}{x^2+y^2}\right); \tag{6}$$

an expression which is equal to zero at every point of the circumference, $x^2 + y^2 = r^2$.

The velocity components u_0, v_0, obtained by differentiating (6), are identical with those given by equations (4) and (5).

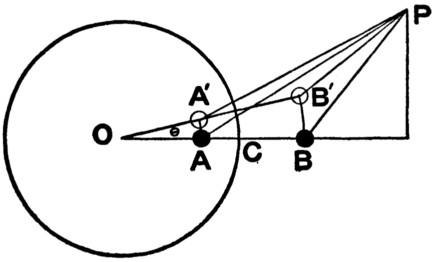

FIG. 5.

5. In Figure 5 let $OA = OA' = a$, $OB = OB' = b$, and $OA \cdot OB = OA' \cdot OB' = \overline{OC}^2 = r^2$, and let there be at A and B sources of strength m, and at A' and B' sinks of equal strength m; then one of the lines of flow due to this combination of sources and sinks in an

indefinitely extended plate will consist in part of the circumference of which OC is a radius. This shall be called the circumference C.

The value, at P, the co-ordinates of which are x and y, of the flow function is

$$\begin{aligned}\psi_P &= -m\left\{\tan^{-1}\frac{y-a\sin\theta}{x-a\cos\theta}+\tan^{-1}\frac{y-b\sin\theta}{x-b\cos\theta}-\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x-b}\right\}\\ &= -m\left\{\tan^{-1}\frac{(x-a)\,(y-a\sin\theta)-y\,(x-a\cos\theta)}{(x-a)\,(x-a\cos\theta)+y\,(y-a\sin\theta)}\right.\\ &\qquad\left.+\tan^{-1}\frac{(x-b)\,(y-b\sin\theta)-y\,(x-b\cos\theta)}{(x-b)\,(x-b\cos\theta)+y\,(y-b\sin\theta)}\right\}.\end{aligned}$$

If now we let θ approach zero as a limit and m increase in such a way that the product of m and the straight line $\overline{AA'}$ shall always be equal to the constant μ, we have

$$\psi_0=\underset{\theta\doteq 0}{\text{Limit}}\ \psi=+\mu\left\{\frac{x-a}{(x-a)^2+y^2}+\frac{r^2}{a^2}\cdot\frac{x-\frac{r^2}{a}}{\left(x-\frac{r^2}{a}\right)^2+y^2}\right\},\quad(7)$$

the real part of the function

$$w=+\mu\left\{\frac{1}{z-a}+\frac{r^2}{a^2}\cdot\frac{1}{z-\frac{r^2}{a}}\right\}.$$

At every point of C, ψ_0 has the value $-\frac{\mu}{a}$.

The velocity components at P corresponding to ψ_0 are

$$u=-2\,\mu y\left\{\frac{x-a}{[(x-a)^2+y^2]^2}+\frac{r^2}{a^2}\cdot\frac{x-\frac{r^2}{a}}{\left[\left(x-\frac{r^2}{a}\right)^2+y^2\right]^2}\right\};\quad(8)$$

$$v=+\mu\left\{\frac{(x-a)^2-y^2}{[(x-a)^2+y^2]^2}+\frac{r^2}{a^2}\cdot\frac{\left(x-\frac{r^2}{a}\right)^2-y^2}{\left[\left(x-\frac{r^2}{a}\right)^2+y^2\right]^2}\right\}.\quad(9)$$

If a is made to approach zero as a limit, these expressions approach respectively the limits

$$u_0=\frac{-2\,\mu\,x\,y}{[x^2+y^2]^2};\quad(10)$$

$$v_0=\mu\left(\frac{x^2-y^2}{[x^2+y^2]^2}+\frac{1}{r^2}\right).\quad(11)$$

which agree with the expressions (4) and (5) obtained by using different co-ordinate axes.

We infer from (7) that the image of a doublet of strength μ so situated at a point A — which is at a distance a from the centre of a circumference drawn with radius r on a thin plane indefinitely extended plate — that its axis is perpendicular to the radius vector drawn through A, is a doublet, of strength $\mu \frac{r^2}{a^2}$ and axis parallel to that of the first doublet, at the inverse point of A with respect to the circumference.

If we make a equal to r, (7), (8), and (9) give us the flow function and the velocity components inside a circular plate when there is a doublet at some point of the circumference with its axis coincident with the tangent to the plate at the given point. The forms of these expressions might have been inferred from the results given in section 2.

6. It is to be noticed that if in Figure 6 $O'B'$ and OB are parallel lines, and OO', AA', and BB' perpendicular to them, and if there are sources of strength m at O', A, and B, and sinks of equal strength m at O, A', and B', a circumference drawn around O as centre with radius $\sqrt{OA\,.\,OB}$ will form part of a line of flow due to the combination of the sink at O and the sources at A and B, but not one of the lines of flow due to the combination of three doublets obtained by making $O'B'$ approach OB and increasing m at the same time so that $m\,.\,\overline{OO'}$ shall always be equal to the constant μ.

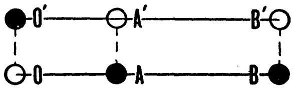

FIG. 6.

7. In Figure 7 let $OA\,.\,OB = OA'\,.\,OB' = OC^2 = r^2$, and $OA = a$, and let the curve AA' make with OM the angle δ. If, then, there are sources of strength m at A' and B', and sinks of strength m at A and B, one of the lines of flow will consist in part of the circumference (C) drawn with radius r about O as centre. If ψ_P is the value at P of the flow function due to this combination, the limit approached by ψ_P as A' is made to approach A along the curve $A'A$, and B' to approach B along the corresponding reciprocal curve $B'B$, will be the value at P of the flow function due to a doublet at A, the axis of which makes the angle δ with the radius drawn through A of the circumference C, and to the image of the doublet in C.

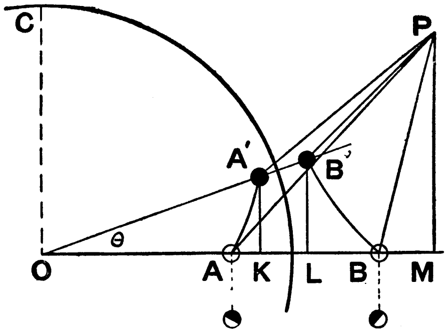

FIG. 7.

$$\psi_P = m\left\{\tan^{-1}\frac{y-\overline{KA'}}{x-a-\overline{AK}} - \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y-\overline{LB'}}{x-\overline{OB}-\overline{BL}} - \tan^{-1}\frac{y}{x-\overline{OB}}\right\}$$

$$= m\left\{\tan^{-1}\frac{y\,.\,\overline{AK}-\overline{KA'}\,(x-a)}{(x-a)^2+y^2-y\,.\,\overline{KA'}-\overline{AK}\,(x-a)}\right.$$

$$\left.+\tan^{-1}\frac{y\,.\,\overline{BL}-\overline{LB'}\,(x-OB)}{(x-OB)^2+y^2-y\,.\,\overline{LB'}-\overline{BL}\,(x-OB)}\right\}.$$

If A' be made to approach A as a limit, we have

$$\text{Limit}\,\frac{\overline{BB'}}{\overline{AA'}} = \frac{r^2}{\overline{OA}^2};\ \text{Limit}\,\frac{\overline{KA'}}{\overline{AA'}} = \sin\delta = \text{Limit}\,\frac{\overline{LB'}}{\overline{BB'}};$$

$$\text{Limit}\,\frac{\overline{AK}}{\overline{AA'}} = \cos\delta = \text{Limit}\,\frac{\overline{LB}}{\overline{BB'}};$$

and, if m be made to change so that $m\,.\,\overline{AA'}$ shall always be equal to the constant μ, Limit $(m\,.\,BB')$ will be equal to the constant, $\frac{r^2}{a^2}\,.\,\mu$.

$$\underset{\overline{AA'}\doteq 0}{\text{Limit}}\,\psi_P = \psi_0 = \mu\left[\underset{\overline{AA'}\doteq 0}{\text{Limit}}\left\{\frac{\tan^{-1}\dfrac{y\,.\,\overline{AK}-\overline{KA'}\,(x-a)}{(x-a)^2+y^2-y\,.\,\overline{KA'}-\overline{AK}(x-a)}}{\overline{AA'}}\right\}\right.$$

$$\left.+\frac{r^2}{a^2}\underset{\overline{BB'}\doteq 0}{\text{Limit}}\left\{\frac{\tan^{-1}\dfrac{y\,.\,\overline{BL}-\overline{LB'}\,(x-\overline{OB})}{(x-\overline{OB})^2+y^2-y\,.\,\overline{LB'}-\overline{BL}(x-\overline{OB})}}{\overline{BB'}}\right\}\right].$$

$$\psi_0 = \mu \,.\, \underset{\overline{AA'} \doteq 0}{\text{Limit}} \left\{ \frac{y \,.\, \dfrac{\overline{AK}}{\overline{AA'}} - \dfrac{\overline{KA'}}{\overline{AA'}}\,(x - a)}{(x-a)^2 + y^2 - y \,.\, \overline{KA'} - \overline{AK}\,(x - a)} \right\}$$

$$+ \frac{\mu\, r^2}{a^2} \underset{\overline{BB'} \doteq 0}{\text{Limit}} \left\{ \frac{y \,.\, \dfrac{\overline{BL}}{\overline{BB'}} - \dfrac{\overline{LB'}}{\overline{BB'}}\,(x - \overline{OB})}{(x - \overline{OB})^2 + y^2 - y \,.\, \overline{LB'} - BL\,(x - \overline{OB})} \right\}$$

$$= \mu \left\{ \frac{y \,.\, \cos\delta - (x - a)\sin\delta}{(x-a)^2 + y^2} - \frac{r^2}{a^2} \,.\, \frac{y \,.\, \cos\delta + \left(x - \dfrac{r^2}{a}\right)\sin\delta}{\left(x - \dfrac{r^2}{a}\right)^2 + y^2} \right\}. \quad (12)$$

It follows from (12) that the image of a doublet of strength μ so situated at a point A — which is at a distance a from the centre, O, of a circumference drawn with radius r on a thin, plane, indefinitely extended plate — that its axis makes the angle δ with the radius drawn through A, is a doublet of strength $\frac{\mu\, r^2}{a^2}$ and axis making an angle of 180° — δ with the line OA, at the inverse point of A with respect to the circumference.

8. It was long ago noticed [*] that the functions

$$\log z,\ \frac{1}{z},\ -\frac{1}{z^2},\ \frac{2}{z^3},\ -\frac{6}{z^4},\ \ldots.,$$

each of which is the derivative with respect to z of the one which precedes it, yield a series of pairs of conjugate functions which represent in order the velocity potential functions and the flow functions due to a source at the origin, to a doublet at the origin, to a quadruplet [†] at the origin, to an octuplet at the origin, and so on.

I will use the word *motor* to denote in general a source, a sink, a doublet, a quadruplet, or any other combination of sources or sinks at a single point. The strengths of two motors of the same kind shall be in the ratio of p to q if, when they are similarly placed at the same point successively, they give rise to velocity potential functions and flow functions which have values in the ratio of p to q at every point. The unit motor of any kind and the direction of its axis, if it has any, may be chosen at pleasure.

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[*] See, for instance, Klein's Lectures on the Potential, or his paper, "Ueber Riemann's Theorie der Algebraischen Functionen und ihrer Integrale."

[†] A quadruplet is formed of two equal and opposite doublets in the same manner that a doublet is formed out of a source and an equal sink. An octuplet is formed in a similar way of two equal and opposite quadruplets, and so on.

Let there be in a plane any distribution, A, of n motors, either all of the same kind or of different kinds, of strengths respectively equal to $m_1, m_2, m_3, \ldots . m_n$.

Let there be also a system, B, of n motors, each respectively equal but opposite to a corresponding motor of A, and so situated in the plane that every motor of A could be superposed upon its equivalent in B by moving A parallel to itself through a distance δ in a direction making an angle α with the axis of x.

Let $u = f(x, y)$ and $v = \chi(x, y)$ be the values at the point (x, y) of the velocity components due to the system A, and let $\phi(x, y)$ and $\psi(x, y)$ be the corresponding values of the velocity potential function and the flow function.

It is evident that the velocity components due to B have values at (x, y) equal but opposite to those at the point $(x - \delta \cos \alpha, y - \delta \sin \alpha)$ of the corresponding velocity components due to A. We may, therefore, take for the values at any point (x, y) of the velocity potential and flow functions due to A and B existing together, the expressions

$$\phi(x, y) - \phi(x - \delta \cos \alpha, y - \delta \sin \alpha)\,; \qquad (13)$$

$$\psi(x, y) - \psi(x - \delta \cos \alpha, y - \delta \sin \alpha). \qquad (14)$$

If now B be made to approach A, by decreasing δ and keeping α constant, and if the strength of each of A's motors be made to grow so as to keep the product of itself and δ constant, the expressions just given approach as limits the velocity potential function and the flow function which A would yield if every one of its motors were *doubled up* with an equal but opposite motor approaching it from a direction which makes an angle α with the axis of x.

If $m_1 \delta = \mu_1$, $m_2 \delta = \mu_2$, $m_3 \delta = \mu_3$, etc., where the μ's are constant, the limits of the expressions (13) and (14) are the values obtained by changing every m into its corresponding μ in

$$\underset{\delta \doteq 0}{\text{Limit}} \left(\frac{\Delta_x \phi + \Delta_y \phi}{\delta}\right) \text{ and } \underset{\delta = 0}{\text{Limit}} \left(\frac{\Delta_x \psi + \Delta_y \psi}{\delta}\right)\,;$$

that is, in
$$\cos \alpha \,.\, D_x \phi + \sin \alpha \,.\, D_y \phi, \qquad (15)$$

and
$$\cos \alpha \,.\, D_x \psi + \sin \alpha \,.\, D_y \psi. \qquad (16)$$

If $w = f(z) = \phi(x, y) + i \,.\, \psi(x, y)$, the function of z which corresponds to the new velocity potential function and flow function may be found by changing every m into its corresponding μ in

$$-(\cos \alpha + i \,.\, \sin \alpha) \,.\, D_z w. \qquad (17)$$

9. The equation $\frac{\sin n\,\theta}{r^n} = c$ represents a family of closed curves each of which is made up of $2\,n$ equal mutually distinct loops symmetrically situated about the origin; each curve passes n times through the origin and each loop is tangent there to two straight lines which include an angle of $\frac{\pi}{n}$. The equation $\frac{\cos n\,\theta}{r^n} = c$ represents the same family of curves turned around the origin through an angle $\frac{\pi}{2\,n}$.

The motor which gives rise to the flow function $\frac{\lambda \sin 2\,\theta}{r^2}$ may be denoted by the symbol shown in the upper part of Figure 8, in which the black portions show the directions in which fluid flows out from the point at which the motor is situated, and the open portions the directions from which fluid flows towards this point. The motor which corresponds to the flow function $\frac{\lambda \sin n\,\theta}{r^n}$ may be represented similarly by a circle drawn about the origin, and properly divided into n shaded and n unshaded sectors.

Let the motor A, which corresponds to the flow function $\psi_A = \frac{\sin n\,\theta}{r^n}$, be doubled up with an equal opposite motor which approaches it from a direction making an angle a with the axis of x. The flow function ψ_B, due to the resulting motor B, is given, apart from a constant factor, by the equation

$$\begin{aligned}\psi_B &= D_x\,\psi_A \cdot \cos a + D_y\,\psi_A \cdot \sin a \\ &= D_r\,\psi_A \cdot \cos(\theta - a) - \frac{D_\theta\,\psi_A}{r} \cdot \sin(\theta - a) \\ &= -\frac{n \cdot \sin\left[(n+1)\,\theta - a\right]}{r^{n+1}}. \qquad (18)\end{aligned}$$

Referred to a new initial line drawn in a direction making an angle $\frac{a}{n+1}$ with the old, this is

$$\psi_B = \frac{n \cdot \sin(n+1)\,\theta}{r^{n+1}}; \qquad (19)$$

and it is evident that the orientation of B's axis, but not B's character, depends upon a.

For instance, the flow function due to a unit doublet at the origin with axis coincident with the axis of x is

$$\psi_A = \frac{\mu\,y}{x^2 + y^2}.$$

If this motor be doubled up with its negative approaching from the positive part of the axis of x, the resultant quadruplet (see Fig. 8) corresponds to the flow function

$$\psi_B = \frac{-2xy}{(x^2+y^2)^2}.$$

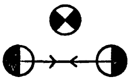

FIG. 8.

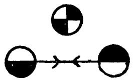

FIG. 9.

If, on the other hand, it be doubled up with an equal opposite doublet (see Fig. 9) approaching it from the positive part of the axis of y, the resultant quadruplet corresponds to the flow function

$$\psi_B = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

One of these quadruplets is evidently equivalent to the other turned through an angle of 45°.

10. Besides the simple quadruplets obtained by combining two equal and opposite doublets just as a doublet is formed from a source and a sink, other combinations of two doublets sometimes appear when one attempts to find the image of a simple motor in the circumference of a circle in its plane.

In one common case, a fixed doublet D_1 of strength μ is approached by another doublet D_2, the axis of which remains always parallel but opposite in direction to that of D_1. The path of D_2 is a straight line, but its strength is equal to $\mu f(\delta)$, where δ is the distance between the two doublets at any time, and $f(\delta)$ is a finite, continuous, and single-valued function such that $f(0) = 1$. The product of μ and δ is kept always equal to a constant, λ.

The flow function due to a doublet of strength μ at the origin with axis coincident with the axis of x is $\frac{\mu y}{x^2+y^2}$, and the flow function due to a reversed doublet of strength $\mu f(\delta)$ at the point $(x - \delta . \cos \alpha,\ y - \delta . \sin \alpha)$

is
$$\frac{-\mu . (y - \delta \sin \alpha) . f(\delta)}{(x - \delta . \cos \alpha)^2 + (y - \delta . \sin \alpha)^2}.$$

The flow function due to the two doublets existing together is

$$\frac{\lambda}{\delta}\left\{\frac{y}{x^2+y^2} - \frac{(y - \delta . \sin \alpha)f(\delta)}{(x - \delta . \cos \alpha)^2 + (y - \delta . \sin \alpha)^2}\right\},$$

and the limit of this as δ approaches zero is

$$\frac{\lambda\,\{(x^2-y^2)\sin\alpha - 2\,xy\cos\alpha - y\,(x^2+y^2)f'(0)\}}{(x^2+y^2)^2}; \qquad (20)$$

and this is equivalent to a simple quadruplet superposed upon a doublet of strength $-\lambda\,.f'(0)$ at the origin.

In general, let $\mu\,.\,\psi(x, y)$ be the flow function due to a fixed motor of strength μ, and of any kind; and let this motor be approached (keeping $\mu\,\delta$ constantly equal to λ) by an opposite motor of strength $\mu\,.f(\delta)$ from a direction making an angle α with the axis of x. The value at the point (x, y) of the flow function due to this second motor is the same as the value at the point $(x-\delta\,.\cos\alpha,\ y-\delta\,.\sin\alpha)$ of the flow function due to the first motor, so that the flow function due to the two motors existing together is

$$\frac{\lambda}{\delta}\,[\psi(x, y) - f(\delta)\,.\,\psi(x-\delta\cos\alpha,\ y-\delta\sin\alpha)],$$

and the limit approached by this as δ approaches zero is

$$\lambda\,[D_x\psi\,.\cos\alpha + D_y\psi\,.\sin\alpha - \psi(x, y)\,.f'(0)]. \qquad (21)$$

11. Let there be two equal opposite motors of the same kind at the points A and B in a plane equally distant from an origin, O, and let the axis of the motor at A make with the straight line OA the same angle that the axis of the motor at B would make with OB if its flow function were the negative of what it really is. If now OB be rotated about O so as to approach OA as a limit, and if the product of the strength of one of the motors by its distance from the other be kept equal to a constant, λ, we get as a limit a kind of motor which sometimes appears in practical problems.

Let $\mu\,.\,R(r, \theta)$ and $\mu\,.\,\Theta(r, \theta)$ be the values at the point P, the co-ordinates of which are r and θ, of the velocity components due to the motor at A, taken respectively parallel and perpendicular to the radius vector, OP. If the angle AOB (see Fig. 10) be α, the values of the velocity components at P due to the motor at B are equal but opposite in sign to values at the point P', the co-ordinates of which are r and $\theta-\alpha$, of the velocity components due to the motor at A. Hence the motors at A and B together cause at P the velocity components

$$\mu\,.\,R(r, \theta) - \mu\,.\,R(r, \theta-\alpha),$$

$$\mu\,.\,\Theta(r, \theta) - \mu\,.\,\Theta(r, \theta-\alpha).$$

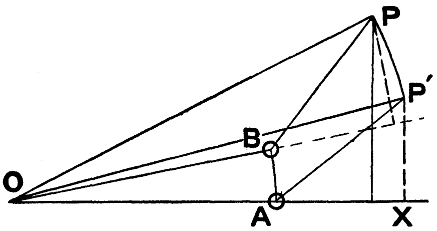

FIG. 10.

The limits of these expressions as a is made to approach zero, while $\mu . a . \overline{OA} = \lambda$, are

$$\frac{\lambda . D_\theta R}{\overline{OA}} \text{ and } \frac{\lambda . D_\theta \Theta}{\overline{OA}};$$

so that the flow function due to the newly found motor is obtained by changing μ to λ in the product of the reciprocal of $\overline{OA}$ and the derivative with respect to θ of the flow function due to the original motor at A alone; that is, in

$$\frac{x . D_y \psi - y . D_x \psi}{\overline{OA}}, \tag{22}$$

where $\mu R = \frac{D_\theta \psi}{r}$ and $\mu \Theta = - D_r \psi$.

A single doublet on the axis of x, at a distance a from the origin, with its axis perpendicular to the axis of x, yields, when combined with another similar motor in the way described in this section, the flow function

$$\psi = \frac{\lambda y \, (x^2 + y^2 - a^2)}{a \, [(x - a)^2 + y^2]^2}. \tag{23}$$

This is equal to zero on the axis of x and on the circumference of the circle drawn with the origin as centre so as to pass through the motor. It gives, therefore, a case of flow in a semicircular disk.

The flow function just found is the real factor of the imaginary part of the function

$$w = - \frac{\lambda z}{a \, (z - a)^2} = - \frac{\lambda}{a} \left\{ \frac{1}{z - a} + \frac{a}{(z - a)^2} \right\}. \tag{24}$$

The motor which corresponds to this flow function may be looked upon, therefore, as formed by superposing a doublet upon a quadruplet.

The image of such a motor (M) as this in a circumference drawn in its own plane with O as centre is a motor of the same kind as M, but of strength $\frac{r^2}{a^2}$ times as great as *M*'s strength, situated at the inverse point with respect to the circumference of the point where *M* lies.

The flow function in this case is easily found, by a method analogous to that used in Section 5, to be

$$\frac{\lambda y}{a}\left\{\frac{x^2+y^2-a^2}{[(x-a^2)+y^2]^2}+\frac{r^2}{a^2}\cdot\frac{x^2+y^2-\frac{r^4}{a^2}}{\left[\left(x-\frac{r^2}{a}\right)^2+y^2\right]^2}\right\}. \tag{25}$$

The combination in the same manner of two pairs of motors of the kind just mentioned yields still another case of flow inside a circular disk due to a certain motor at a point Q, the co-ordinates of which are a and 0, and another motor at the inverse point of Q with reference to the circumference. The flow function in this case is

$$\psi=\frac{\lambda}{a^2}\left[\frac{x^2+y^2-a^2}{[(x-a)^2+y^2]^3}\{x(x-a)^2+y^2(x-4a)\}\right.$$
$$\left.+\frac{r^2}{a^2}\cdot\frac{x^2+y^2-\frac{r^4}{a^2}}{\left[\left(x-\frac{r^2}{a}\right)^2+y^2\right]^3}\left\{x\left(x-\frac{r^2}{a}\right)^2+y^2\left(x-\frac{4r^2}{a}\right)\right\}\right], \tag{26}$$

and this process might be carried on indefinitely.

12. The image of a simple quadruplet in the circumference of a circle drawn in its own plane is not another simple quadruplet, but a more complex motor.

On a straight line which passes through the centre of a circumference of radius r drawn in a plane, and which is taken for axis of x, let there be two equal and opposite doublets of strength μ with axis coincident with the line, at points distant respectively a and $a+\Delta a$ from the centre of the circumference. Let there be also on this line the two images of these doublets with respect to the circumference.

The value at the point (x, y) of the flow function due to these four doublets is

$$\mu y\left\{\frac{1}{(x-a)^2+y^2}-\frac{1}{(x-a-\Delta a)^2+y^2}+\right.$$
$$\left.\frac{r^2}{(a+\Delta a^2)\left[\left(x-\frac{r^2}{a+\Delta a}\right)^2+y^2\right]}-\frac{r^2}{a^2\left[\left(x-\frac{r^2}{a}\right)^2+y^2\right]}\right\};$$

and if Δa be made to approach zero, and μ to increase so that $\mu \,.\, \Delta a$ always has the constant value λ, this function approaches as its limit

$$\lambda y \left\{ \frac{2\,(a - x)}{[(x - a)^2 + y^2]^2} + \frac{r^2}{a^4} \, \frac{2\,r^2\,x - 2\,a\,(x^2 + y^2)}{\left[\left(x - \frac{r^2}{a}\right)^2 + y^2\right]^2} \right\}. \qquad (27)$$

If the axis of a simple quadruplet be a diameter drawn through the centre of the shaded portions of the symbol which represents the motor, the expression just found gives, with proper algebraic sign, the flow function due to a simple quadruplet in a circular disk with axis parallel or perpendicular to the radius drawn through the point at which the motor is situated.

The flow function, which vanishes at every point on the circumference, is the real factor of the imaginary part of the function

$$w = -\lambda \left\{ \frac{1}{(z - a)^2} + \frac{r^2}{a^4} \cdot \frac{r^2 - 2\,a\,z}{\left(z - \frac{r^2}{a}\right)^2} \right\}$$

$$= -\lambda \left\{ \frac{1}{(z - a)^2} - \frac{r^4}{a^4} \cdot \frac{1}{\left(z - \frac{r^2}{a}\right)^2} - \frac{2\,r^2}{a^3} \cdot \frac{1}{z - \frac{r^2}{a}} \right\}. \qquad (28)$$

The image of the simple quadruplet in question is made up, therefore, of a simple quadruplet and a doublet existing together at the inverse point, with respect to the circumference, of the point where the original quadruplet is situated.

13. If the axes of the doublets mentioned in the last section had been perpendicular to the axis of x instead of coincident with it, the flow function due to the four doublets would have been

$$\mu \left\{ \frac{x - a}{(x - a)^2 + y^2} - \frac{x - a - \Delta a}{(x - a - \Delta a)^2 + y^2} \right.$$

$$\left. + \frac{r^2}{a^2} \cdot \frac{x - \frac{r^2}{a}}{\left(x - \frac{r^2}{a}\right)^2 + y^2} - \frac{r^2}{(a + \Delta a)^2} \cdot \frac{x - \frac{r^2}{a + \Delta a}}{\left(x - \frac{r^2}{a + \Delta a}\right)^2 + y^2} \right\}.$$

The limit of this, as Δa aproaches zero and $\mu\,\Delta a$ is kept equal to the constant λ, is

$$+\lambda\left\{\frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2}-\frac{r^4}{a^4}\cdot\frac{1}{\left(x-\frac{r^2}{a}\right)^2+y^2}\right.$$

$$\left.+\frac{2\,r^3}{a^3}\cdot\frac{\left(x-\frac{r^2}{a}\right)\left(x^2+y^2-\frac{r^2\,x}{a}\right)}{\left[\left(x-\frac{r^2}{a}\right)^2+y^2\right]^2}\right\}. \qquad (29)$$

This has the value $-\frac{\lambda}{a^2}$ at all points of the circumference, and gives us the flow function inside a circular disk in which there is a simple quadruplet with axis making an angle of 45° with the radius drawn through it.

It is to be noticed that (29) is the real part of the function

$$w=-\lambda\left\{\frac{1}{(z-a)^2}-\frac{r^4}{a^4}\cdot\frac{1}{\left(z-\frac{r^2}{a}\right)^2}-\frac{2\,r^2}{a^3}\cdot\frac{1}{z-\frac{r^2}{a}}\right\}, \qquad (30)$$

which appears in equation (28).

14. The flow function due to a simple quadruplet at the centre of a circular disk of radius r may be found by putting a equal to zero in (27). It is

$$\psi=-2\,\lambda\,x\,y\left\{\frac{1}{(x^2+y^2)^2}-\frac{1}{r^4}\right\},$$

and the corresponding w is $\frac{1}{z^2}+\frac{z^2}{r^4}$.

In general, consider the function

$$w=\frac{1}{z^n}+\frac{z^n}{r^{2n}}=\frac{r^{2n}+\rho^{2n}\,(\cos 2\,n\,\theta+i\,.\,\sin 2\,n\,\theta)}{r^{2n}\,.\,\rho^n\,(\cos n\,\theta+i\,.\,\sin n\,\theta)}$$

$$=\frac{r^{2n}\,.\,\cos n\,\theta+\rho^{2n}\,(\cos 2\,n\,\theta\,.\,\cos n\,\theta+\sin 2\,n\,\theta\,.\,\sin n\,\theta}{r^{2n}\,.\,\rho^n}$$

$$+\frac{i\,\{\rho^{2n}\,(\sin 2\,n\,\theta\,.\,\cos n\,\theta-\cos 2\,n\,\theta\,.\,\sin n\,\theta)-r^{2n}\sin n\,\theta\}}{r^{2n}\,\rho^n}$$

$$=\frac{r^{2n}+\rho^{2n}}{r^{2n}\,.\,\rho^n}\cos n\,\theta+\frac{i\,(\rho^{2n}-r^{2n})}{r^{2n}\,\rho^n}\sin n\,\theta. \qquad (31)$$

The function

$$\psi = \frac{(\rho^{2n} - r^{2n}) \sin n\theta}{r^{2n}\rho^n} \tag{32}$$

is constant along the circumference drawn about the origin with radius r, and is evidently the flow function due to a simple 2^{n+1}-plet at the centre of a circular disk.

15. The rational algebraic function $\frac{\chi(z)}{F(z)}$, where the numerator and denominator are algebraic polynomials involving only integral powers of z, can be put by simple division into the form of a rational algebraic polynomial in z plus the proper fraction, $\frac{f(z)}{F(z)}$, where the numerator is of degree at least one lower than the denominator. This proper fraction can be represented in the usual way as the sum of a series of fractions the numerators of which are in general complex constants and the denominators of the form $[z - (a + bi)]^\kappa$.

The real part of $\frac{f(z)}{F(z)}$ is therefore the velocity potential function of a collection of motors such that each has for its own velocity potential function the real part of a function of the form

$$\frac{\alpha + \beta i}{(z - (a + bi))^\kappa} \quad \text{or} \quad \frac{\mu(\cos\delta + i\sin\delta)}{(z - (a + bi))^\kappa}. \tag{33}$$

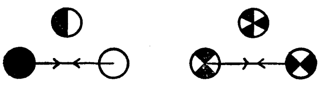

Fig. 11.

Let us take as the axis of a simple multiplet a line which, drawn as a diameter through the symbol which represents the motor, bisects the black portion nearest the axis of x. If this line goes through two black portions, the sense in which the direction of the axis is taken is of no consequence. If it goes through one black sector and one unshaded sector of the symbol, the direction shall be *from* the unshaded part *to* the black part. The simple motor, for instance, which has for its flow function the real factor of the imaginary part of $-\frac{1}{z^\kappa}$, has its axis coincident with the axis of x.

The values at the point (x, y) of the velocity components due to any motor at the point (a, b) must be equal respectively to the two velocity components at the point $(x - a, y - b)$ due to the same motor at the origin with axis parallel to its old direction.

If $\phi(x, y)$, $\psi(x, y)$, are the velocity potential and flow functions due to the motor at the origin, $\phi(x - a, y - b)$, $\psi(x - a, y - b)$, yield velocity components for the case of the same motor at (a, b) which satisfy the condition stated above.

If $$f(z) = \phi(x, y) + i\,\psi(x, y), \text{ however,}$$

$$f(z - a - b\,i) = \phi(x - a, y - b) + i \,.\, \psi(x - a, y - b),$$

and, as a particular case, $\dfrac{1}{[z - (a + b\,i)]^{\kappa}}$ and $\dfrac{1}{z^{\kappa}}$ correspond to two equal and similarly placed motors, one at the point (a, b), the other at the origin.

Let us now compare the motors the flow functions of which are the real factors of the imaginary parts of

$$\frac{1}{z^{\kappa}} \quad \text{and} \quad \frac{\alpha + \beta\,i}{z^{\kappa}}$$

respectively; that is, of

$$\frac{\text{ces}\,\kappa\,\theta}{\rho^{\kappa}} \quad \text{and} \quad \frac{\mu\,\text{cis}\,\delta \,.\, \text{ces}\,\kappa\,\theta}{\rho^{\kappa}},$$

where $z = \rho(\cos\theta + i \,.\, \sin\theta)$, $\alpha + \beta\,i = \mu\,(\cos\delta + i \,.\, \sin\delta)$, and where $(\cos\delta + i\sin\delta)$ and $(\cos\theta - i \,.\, \sin\theta)$ are written $\text{cis}\,\delta$ and $\text{ces}\,\theta$ respectively.

The two flow functions are

$$\frac{-\sin\kappa\,\theta}{\rho^{\kappa}} \quad \text{and} \quad -\frac{\mu \,.\, \sin\kappa\left(\theta - \dfrac{\delta}{\kappa}\right)}{\rho^{\kappa}};$$

and the values at the point (r, θ) of the velocity components respectively parallel and perpendicular to the radius vector drawn from the origin are

$$-\frac{\kappa \,.\, \cos\kappa\,\theta}{\rho^{\kappa+1}} \quad \text{and} \quad \frac{\kappa \,.\, \sin\kappa\,\theta}{\rho^{\kappa+1}}$$

in the case of the first motor, and

$$-\frac{\mu\,\kappa\,.\,\cos\kappa\left(\theta-\frac{\delta}{\kappa}\right)}{\rho^{\kappa+1}}\quad\text{and}\quad\frac{\mu\,\kappa\,.\,\sin\kappa\left(\theta-\frac{\delta}{\kappa}\right)}{\rho^{\kappa+1}}$$

in the case of the second motor. The second motor gives at every point velocity components which the first motor gives at a point equally distant from the origin, but differing in vectorial angle from the first point by $\frac{\delta}{\kappa}$. The second motor is then equivalent to the first turned through the angle $\frac{\delta}{\kappa}$.

If, then, the simple motor which corresponds to the function $\frac{\mu}{z^\kappa}$ were transferred parallel to itself to the point (a, b) and then rotated counter-clockwise through the angle $\frac{\delta}{\kappa}$, it would correspond to the function $\frac{\alpha+\beta i}{[z-(a+b\,i)]^\kappa}$. It is to be noticed that a rotation of the motor corresponding to $\frac{\mu}{z^\kappa}$ through the angle $\frac{2\,\pi}{\kappa}$ would by reason of symmetry give the same motor, and multiplying $\frac{\mu}{z^\kappa}$ by i would turn the symbol representing the motor through half the angle which corresponds to a black sector.* The rational algebraic proper fraction $\frac{f(z)}{F(z)}$ corresponds then, in general, to a distribution of variously oriented simple multiplets, and the function $A\,z+\frac{f(z)}{F(z)}$, which frequently appears in two dimensional problems in magnetism, to a distribution of such multiplets in a uniform field.†

16. The real parts of the functions $-\frac{\lambda}{z^\kappa}$, $-\frac{\mu\text{ cis }\kappa\,a}{z^\kappa}$, are the velocity potential functions which correspond to two simple multiplets of the κth order, L and M, of strengths respectively equal to λ and μ. Both L and M are situated at the origin. L's axis coincides with the axis of x, but M's axis makes with the axis of x the angle a.

If we superpose L upon M, we get

$$w=\frac{-(\lambda+\mu\,.\,\cos\kappa\,a+i\,.\,\mu\,.\,\sin\kappa\,a)}{z^\kappa}$$

$$=-\frac{\sigma\,.\,\text{cis}\,\tau}{z^\kappa}, \tag{34}$$

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* This explains the identity of equations (28) and (30).

† See, for instance, Maxwell's Treatise on Electricity and Magnetism, Vol. II. Figs. XV. and XVI.

where

$$\sigma^2 = \lambda^2 + \mu^2 + 2\,\lambda\,\mu\cos\kappa\,a, \tag{35}$$

$$\tau = \sin^{-1}\left\{\frac{\mu\sin\kappa\,a}{\sqrt{\lambda^2 + \mu^2 + 2\,\lambda\,\mu\cos\kappa\,a}}\right\}. \tag{36}$$

This is a motor of the same kind as L and M. Its strength is σ, and its axis makes with the axis of x the angle $\frac{\tau}{\kappa}$. In the case of doublets, that is, when $\kappa = 1$, this result may be put into a simple form. If two doublets L and M exist together at a point O, and if the directions of the two straight lines OA, OB, show the directions of the axes of L and M respectively, and the lengths of OA and OB the strengths of L and M on some convenient scale, then the direction of the axis of the result of L and M will be given by the directions and the strengths of the resultant by the length of the diagonal of the parallelogram of which OA and OB are adjacent sides. Doublets then can be compounded and resolved by compounding and resolving their axes like forces or velocities.

17. Let there be any motor M at a distance a from the centre of a circumference of radius r drawn in a plane, and let the radius drawn through M be taken as axis of x. Let $\mu \,.\, \chi(x, y, a)$ be the flow function due to M, and its image in the circumference, existing together. If another motor B, of the same kind and strength as A but of opposite sign, exist at a point on the axis of x at a distance $a + \Delta a$ from the centre, the flow function due to B and its image together is

$$-\,\mu \,.\, \chi(x, y, a + \Delta\,a).$$

The flow function due to A, B, and their two images existing together is, therefore, $-\,\mu \,.\, \Delta_a\,\chi(x, y, a)$.

If now Δa be made to approach zero and μ to increase so that $\mu \,.\, \Delta a$ is equal to the constant λ, we get as the flow functions due to the resulting new motor at A and its image in the circumference

$$\psi = -\,\lambda \,.\, D_a\,\chi(x, y, a). \tag{37}$$

$F(z, a)$, of which ψ is the real factor of the imaginary part, is connected with $f(z, a)$, of which χ is the real factor of the imaginary part by the equation

$$F(z, a) = -\,\lambda \,.\, D_a f(z, a). \tag{38}$$

If, for instance, we apply (38) to

$$w = \mp \mu \left\{ \frac{1}{z - a} + \frac{r^2}{a^2} \cdot \frac{1}{z - \frac{r^2}{a}} \right\},$$

which yields (5), we get

$$\pm \lambda \left\{ \frac{1}{(z - a)^2} + \frac{r^2}{a^4} \cdot \frac{r^2 - 2\,a^2}{\left(z - \frac{r^2}{a}\right)^2} \right\},$$

which, after reduction, gives (30).

By means of (38) and other similar equations, a great number of cases of flow inside circular disks may be found.

Jefferson Physical Laboratory,
Cambridge, Mass., August 18, 1891.